EXPANSION OF EIGENVALUES OF SCHRÖDINGER-TYPE OPERATORS ON ONE DIMENSIONAL LATTICES

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Abstract
We study the discrete spectrum of a wide class of discrete Schrödinger operators $\widehat{H}_\mu = \widehat{H}_0 + \mu\widehat{V}$ on the one dimensional lattice $\mathbb{Z}$, where $\widehat{H}_0$ is a Laurent-Toeplitz-type convolution operator, $\widehat{V}$ is a finite rank potential and $\mu \geq 0$ is the coupling constant. We study the asymptotics of eigenvalues of $\widehat{H}_\mu$ lying above the essential spectrum as $\mu \to +\infty$.

Key words and phrases: dispersion relations, eigenvalues, essential spectrum.

1. Discrete Schrödinger operator

Let $\mathbb{Z}$ be the one dimensional cubical lattice and $\ell^2(\mathbb{Z})$ be the Hilbert space of square-summable functions on $\mathbb{Z}$. We denote by $\mathbb{T} := (-\pi, \pi]$, the one dimensional torus, the dual group of $\mathbb{Z}$. Let $L^2(\mathbb{T})$ be the Hilbert space of square-integrable functions on $\mathbb{T}$.

In the coordinate space representation the energy operator $\widehat{H}_\mu$ of a one-particle system on the one-dimensional lattice $\mathbb{Z}$ with a potential field $\hat{v}$ is defined as

$$\widehat{H}_\mu := \widehat{H}_0 + \mu\widehat{V}, \qquad \mu \geq 0, \qquad (1)$$

where the free energy operator $\widehat{H}_0$ is a Laurent-Toeplitz-type operator in $\ell^2(\mathbb{Z})$

$$\widehat{H}_0\hat{f}(x) = \sum_{y\in\mathbb{Z}} \hat{\varepsilon}(x-y)\hat{f}(y), \qquad \hat{f} \in \ell^2(\mathbb{Z}),$$

given by given by a Hopping matrix $\hat{\varepsilon} \in \ell^1(\mathbb{Z})$ of the particle which satisfies $\hat{\varepsilon}(x) = \overline{\hat{\varepsilon}(-x)}$ for all $x \in \mathbb{Z}$, and the potential energy operator is the multiplication in $\ell^2(\mathbb{Z})$ by the function

$$\hat{v}(x) = \begin{cases} a, & \text{if} x = 0, \\ b, & \text{if} |x| = 1, \\ 0, & \text{if} |x| > 1, \end{cases}$$

where $a, b \in \mathbb{R}\backslash\{0\}$.

In the momentum space representation, the operator acts in $L^2(\mathbb{T})$ by

$$H_\mu = \mathcal{F}\widehat{H}_0\mathcal{F}^* + \mu\mathcal{F}\widehat{V}\mathcal{F}^* = H_0 + \mu V,$$

where $\mathcal{F}$ and $\mathcal{F}^*$ is the standard Fourier transform and its inverse. The free hamiltonian H_0 is the multiplication operator in $L^2(\mathbb{T})$ by the function $\mathfrak{e} := \sqrt{2\pi}\mathcal{F}\hat{\varepsilon}$ so-called the dispersion relation of the particle and the potential V acts on $L^2(\mathbb{T})$ as a convolution operator

$$Vf(p) = \frac{1}{2\pi}\int_{\mathbb{T}} (a + 2b\sum_{i=1}^{2}\cos(p_i - q_i))f(q)dq.$$

Since V is compact, by Weyl's Theorem [6, Theorem XIII.14],

$$\sigma_{ess}(H_\mu) = \sigma(H_0) = [\min \mathfrak{e}, \max \mathfrak{e}] = [\mathfrak{e}_{min}, \mathfrak{e}_{max}].$$

In what follows we always assume:

Hypothesis 1. The dispersion relation $\mathfrak{e}$ is a real-valued even function and having a non-degenerate unique maximum at $\pi \in \mathbb{T}$. Moreover, $\mathfrak{e}$ is analytic near π.

2. Main results

Let $L^{2,e}(\mathbb{T})$ and $L^{2,o}(\mathbb{T})$ be the subspaces of essentially even and essentially odd functions in $L^2(\mathbb{T})$. Recall that

$$L^2(\mathbb{T}) = L^{2,e}(\mathbb{T}) \oplus L^{2,o}(\mathbb{T}).$$

Since $\mathfrak{e}(p)$ is even, each of the subspaces $L^{2,e}(\mathbb{T})$ and $L^{2,o}(\mathbb{T})$ is invariant with respect to H_μ. Thus, we study the discrete spectrum of H_μ separately restricted to these subspaces. Moreover, we study the asymptotics of eigenvalues as $\mu \to +\infty$. The existence of eigenvalues for this operator is explored in this work [4]. In this work, we only define the asymptotic expansions of eigenvalues as $\mu \to +\infty$.

Theorem 1. (a) Let $ab < 0$ and let $E_e(\cdot)$ be the unique eigenvalue of $H_\mu|_{L^{2,e}(\mathbb{T})}$. Then

$$E_e(\mu) = \begin{cases} b\mu + \frac{b}{2\pi}\int_{\mathbb{T}} \cos^2 q\, \mathfrak{e}(q)dq + O(1/\mu) & \text{if } b > 0 > a \\ a\mu + \frac{a}{2\pi}\int_{\mathbb{T}} \mathfrak{e}(q)dq + O(1/\mu) & \text{if } a > 0 > b \end{cases}$$

as $\mu \to +\infty$.

(b) Let $a, b > 0$ and let $E_e^{(1)}(\cdot)$ and $E_e^{(2)}(\cdot)$ be the eigenvalues of $H_\mu|_{L^{2,e}(\mathbb{T})}$. Then

$$\max\{E_e^{(1)}(\mu), E_e^{(2)}(\mu)\} = \max\{\psi_a(\mu), \psi_b(\mu)\} + O(1/\mu)$$

and

$$\min\{E_e^{(1)}(\mu), E_e^{(2)}(\mu)\} = \min\{\psi_a(\mu), \psi_b(\mu)\} + O(1/\mu)$$

as $\mu \to +\infty$, where

$$\psi_a(\mu) := a\mu + \frac{1}{2\pi}\int_{\mathbb{T}} \mathfrak{e}(q)dq, \quad \psi_b(\mu) := b\mu + \frac{1}{2\pi}\int_{\mathbb{T}} \cos^2 q\, \mathfrak{e}(q)dq.$$

(c) Let $b > 0$ and let $E_o(\cdot)$ be the unique eigenvalue of $H_\mu|_{L^{2,o}(\mathbb{T})}$. Then

$$E_o(\mu) = b\mu + \frac{1}{2\pi}\int_{\mathbb{T}} \sin^2 q\, \mathfrak{e}(q)dq + O(1/\mu)$$

as $\mu \to +\infty$.

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