SOME PROPERTIES OF THE SPACE OF PROBABILITY MEASURES

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In this note we consider covariant functors acting in the categorie of compacts, preserving the shapes of infinite compacts, ANR-systems, moving compacts, shape equivalence, homotopy equivalence and $A(N)SR$ properties of compacts. As well as, shape properties of a compact space X consisting of connectedness components 0 of this compact X under the action of covariant functors, are considered. And as we study the shapes equality $ShX = ShY$ of infinite compacts for the space $P(X)$ of probability measures and its subspaces.

For a compact X by $P(X)$ denote the space of probability measures. It is known that for an infinite compact X, this space $P(X)$ is homeomorphic to the Hilbert cube Q. For a natural number $n \in N$ by $P_n(X)$ denote the set of all probability measures with no more than n support, i.e. $P_n(X) = \{\mu \in P(X) : |supp\mu| \le n\}$. The compact $P_n(X)$ is a convex linear combination of Dirac measures in the form

$$\mu = m_1\delta_{x_1} + m_2\delta_{x_2} + ... + m_n\delta_{x_n}, \sum_{i=1}^{n} m_i = 1, m_i \ge 0, x_i \in X,$$

δ_{x_i} – the Dirac measure at a point x_i. By $\delta(X)$ denote the set of all Dirac measures. Recall that the space $P_f(X) \subset P(X)$ consists of all probability measures in the form $\mu = m_1\delta_{x_1} + m_2\delta_{x_2} + ... + m_k\delta_{x_k}$ of finite support, for each of which $m_i \ge \frac{k}{k+1}$ for some i. For a positive integer n put $P_{f,n} \equiv P_f \bigcap P_n$. For a compact X we have $P_{f,n}(X) = \{\mu \in P_f(X) : |supp\mu| \le n\}$; $P_f^C \equiv P_f \bigcap P^C, P_{f,n}^C \equiv P_f \bigcap P_n \bigcap P^C . P_n^C \equiv P^C \bigcap P.$ For the compact X by $P^C(X)$ denote the set of all measures $\mu \in P(X)$ the support of each of which lies in one of the components of the compact X [7].

For a space X by ΩX denote the expansion (partition) of the space X consisting of all the connected components. If $f : X \to Y$ is a continuous mapping, then the continuous mapping $\Omega f : \Omega X \to \Omega Y$ is uniquely determined by condition $\pi_Y \circ f = \Omega f \cdot \pi_X$, where $\pi_Y : Y \to \Omega Y$ and $\pi_X : X \to \Omega X$, i.e. we have the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ \Omega X & \xrightarrow{\Omega f} & \Omega Y \end{array} \qquad (1)$$

Lemma 1. If X is a compact ANR-space, then the map $P^C(\pi_X)$ is a homotopy equivalence.

Theorem 1. Let X be a compact and let $\pi_X : X \to \Omega X$ be a quotient map. Then the mapping $P^C(\pi_X)$ induces a shape equivalence, i.e. $Sh(P^C(X)) = Sh(\Omega X)$.

Definition [5] A normal subfunctor F of the functor P_n is called locally convex if the set $F(\tilde{n})$ is locally convex.

We say that a functor F_1 is a subfunctor (respectively nadfunktorom?) of a functor F_2 if there exists a natural transformation $h : F_1 \to F_2$ that the map $h(X) : F_1(X) \to F_2(X)$ is a monomorphism (epimorphism) for each object X. By exp denote the hyperspace functor of closed subsets. For example, the identity functor Id is a subfunctor of $\exp_n$, where $\exp_n X = \{F \in \exp X : |F| \le n\}$, and the n th degree functor ^{n} is a nadfunktorom of functors $\exp_n$ and SP_G^n. A normal subfunctor F of the functor P_n is uniquely determined by its value $F(n)$ at an n-point space. Note that $P_n(n)$ is the $(n-1)$-dimensional simplex. Any subset of the $(n-1)$-dimensional simplex σ^{n-1} defines a normal subfunctor of the functor P_n if it is invariant under simplicial mappings.

An example of not normal subfunctor of the functor P_n is the functor of probability measures P_n^C whose supports lie in one of components. One of the examples of locally convex subfunctors of P_n, is a functor $SP^n \equiv SP_{S_n}^n$.

Corollary 1. If for compacts X and Y the equality $|\Omega X| = |\Omega Y| = \aleph_0$ holds, then $Sh(P^C(X)) = Sh(P^C(Y))$ and $ShP(X) = ShP(Y)$, where $|Z|$ is the cardinality of a set Z.

By M_Ω we denote the class of all compacts X such that ΩX is metrizable. From corollary it follows that if $X, Y \in M_\Omega$, then ΩX and ΩY have a countable dense set of isolated points [4].

Corollary 2. If $X, Y \in M_\Omega$, then either $Sh(P^C(X)) \ge Sh(P^C(Y))$ or $Sh(P^C(X)) \le Sh(P^C(Y))$. Therefore, if ΩX and ΩY are infinite, then $Sh(P^C(X)) = Sh(P^C(Y))$, i.e. $Sh(P^C(X)) \ge Sh(P^C(Y))$ and
$Sh(P^C(X)) \le Sh(P^C(Y))$.

Remark. In [6] it is shown that the Borsuk's definition of shapes of compacts is equivalent to the shapes of ANR-systems.

Lemma 2. For any compact X we have $|\Omega P_f(X)| = |\Omega X|$.

Let us note that for all $x \in X$ and $y \in X$ between sets $(r_f^{-1})(x)$ and $(r_f^{-1})(y)$ there is a one-one correspondence, i.e. to an arbitrary point

$\mu_x \in (P_f^{-1})(X)$ we assign $\mu_y \in (P_f^x)^{-1}$, where

$$\mu_x = m_0\delta_{x_0} + m_1\delta_{x_1} + \ldots + m_k\delta_{x_k}, \mu_y = m_0\delta_{y_0} + \ldots + m_k\delta_{x_k}.$$

In the case of the infinite compacts X and Y the spaces $P(X)$ and $P(Y)$ are homeomorphic to the Hilbert cube Q. If A and B are Z-sets lying in the compacts $P(X)$ and $P(Y)$, then by Chapman's theorem [1], $ShA = ShB$ if and only if $P(X)\setminus A$ is homeomorphic to $P(Y)\setminus B$. In [5,7] it is shown that the subspaces $F(X)$ and $F(Y)$ are Z-sets in the compacts $P(X)$ and $P(Y)$, where $F = P_f(X), P_{f,n}(X), P_{f,n}^C(X), P_f^C(X)$. Moreover, it was noted that this space X is a strong deformation retract for $F(X)$. So the following is valid.

Theorem 2. For infinite compacts X and Y the following conditions are equivalent:

1. $ShX = ShY$;
2. $P(X)\setminus P_f(X);\ P(Y)\setminus P_f(Y)$;
3. $P(X)\setminus \delta(X);\ P(Y)\setminus \delta(Y)$;
4. $P(X)\setminus F(X);\ P(Y)\setminus F(Y)$, where $F = P_{f,n}^C, P_f^C$.

Theorem 3. Suppose that X and Y are elements of M_Ω, $X \in M_\Omega$ and $Y \in M_\Omega$. Then the following conditions are equivalent:

1. $Sh(\Omega X) = Sh(\Omega Y)$;
2. $P(X)\setminus P^C(X);\ P(Y)\setminus P^C(Y)$.

Theorem 4. Suppose that X and Y are elements of M_Ω. Then $Sh(\Omega X) = Sh(\Omega Y)$ if and only if $ShX = Sh(\Omega X)$.

It is known that from the inequality $ShX \le ShY$ it follows $Sh(\Omega X) \le Sh(\Omega Y)$. In particular, the equality $ShX = ShY$ implies $Sh(\Omega X) = Sh(\Omega Y)$.

Now let $Sh(\Omega X) = Sh(\Omega Y)$. From the fact that the compacts ΩX and ΩY are zero-dimensional and metrizable, and by Mardeschicha Segal theorem [2], ΩX and ΩY are homeomorphic. If for any $y \in \Omega X$ the set $\pi_y^{-1}(y)$ has the trivial shape, then by Theorem 7 [3] we have $ShY = Sh(\Omega X)$; By virtue of the zero-dimensionality and equality $ShY = Sh(\Omega X)$ it follows $Y;\ \Omega X;\ \Omega Y$.

Note that in this case $ShX = ShY$ and $X;\ Y$, i.e. $ShX = Sh(\Omega X)$ is equivalent to $ShX = ShY$.

Corollary 3. A) The space $P^C(X)$ is an ASR if and only if X is connected;

B) $P^C(X)$ is an $ANSR$ if and only if X has finitely many connected components.

Theorem 5. For any infinite zero-dimensional compacts X and Y the followings are true:

a) If $ShX = ShY$, then $P_n(X);\ P_n(Y)$;

b) if $ShX = ShY$, then $P(X)\setminus P_n(X);\ P(Y)\setminus P_n(Y)$;

c) $ShP_n(X) = ShP_n(Y)$ if and only if $P(X)\setminus P_n(X);\ P(Y)\setminus P_n(Y)$;

d) $ShF(X) = ShF(Y)$ if and only if $P(X)\setminus F(X);\ P(Y)\setminus F(Y)$, where F are locally convex subfunctors of the functor P_n;

e) $ShX = ShY$ if and only if $P(X)\setminus \delta(X);\ P(Y)\setminus \delta(Y)$.

Theorem 6. For any infinite zero-dimensional compacts X and Y the following conditions are equivalent:

1. $ShX = ShY$;
2. $ShF(X) = ShF(Y)$, where $F = P_{f,n}, P^C_{f,n}, P_{f,} P^C_f$;
3. $X;\ Y$;
4. $P(X)\setminus F(X);\ P(Y)\setminus F(Y)$;

Theorem 7. For any infinite compacts X and Y we have:

A) if $ShX = ShY$, then $P(X)\setminus P_n(X);\ P(Y)\setminus P_n(Y)$ for any $n \in N$;

B) if $ShX = ShY$, then $P(X)\setminus F(X);\ P(Y)\setminus F(Y)$, where F are locally convex subfunctors of the functors P_n.

Theorem 8. For any infinite compacts $X \in M_\Omega$ and $Y \in M_\Omega$ we have:

A) $ShX = ShY$ if and only if $P(X)\setminus P_n(X);\ P(Y)\setminus P_n(Y)$;

B) $ShX = ShY$ if and only if $P(X)\setminus F(X);\ P(Y)\setminus F(Y)$.

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